HW # 3 Answer Key

1a $f(x) = \begin{cases} 1 & \text{if} \quad -2 < x < -1 \\ -3 & \text{if} \quad 4 < x < 7 \\ 0 & \text{else} \end{cases}$

$$F\{f\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{-1} e^{-i\omega x}\, dx - \frac{3}{\sqrt{2\pi}} \int_{4}^{7} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-i\omega(-1)} - e^{-i\omega(-2)}}{-i\omega} \right) - \frac{3}{\sqrt{2\pi}} \left(\frac{e^{-i\omega 4} - e^{-i\omega 7}}{-i\omega} \right)$$

$$= \frac{e^{i\omega} - e^{2i\omega} - 3e^{-4i\omega} + 3e^{-7i\omega}}{-\sqrt{2\pi}\, i\omega}$$

1b $g(x) = \begin{cases} e^{ix} & \text{if} \quad -1 < x < 1 \\ 0 & \text{else} \end{cases}$

$$F\{g\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ix}\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i(\omega-1)x}\, dx$$

$$= \frac{e^{-i(\omega-1)1} - e^{-i(\omega-1)(-1)}}{\sqrt{2\pi}\,(-i(\omega-1))}$$

1c $h(x) = x e^{-\frac{1}{2}x^2}$

We know from class that

$$F\{e^{-x^2/2}\}(\omega) = e^{-\omega^2/2}.$$

Also, $\frac{d}{dx} e^{-x^2/2} = e^{-x^2/2}(-x)$

$$= -h(x).$$

We have a formula for F.T. of a derivative:

$$F\{df/dx\}(\omega) = i\omega F\{f\}(\omega)$$

So

$$F\{h(x)\}(\omega) = F\{-\tfrac{d}{dx} e^{-x^2/2}\}(\omega)$$
$$= -i\omega F\{e^{-x^2/2}\}(\omega)$$
$$= -i\omega e^{-\omega^2/2}.$$

2 When one attempts to calculate $F\{\sin(x)\}$, he gets an integral that is undefined:

$$F\{\sin(x)\} \text{ “}=\text{” } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(x) e^{-i\omega x}\, dx$$

$$\text{“}=\text{” } \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}-e^{-ix}}{2i} e^{-i\omega x}\, dx$$

$$\text{“}=\text{” } \frac{1}{\sqrt{2\pi}\,2i} \lim_{R\to\infty} \underbrace{\frac{e^{-i(\omega-1)R}-e^{-i(\omega-1)(-R)}}{-i(\omega-1)} - \frac{e^{-i(\omega+1)R}-e^{-i(\omega+1)(-R)}}{-i(\omega+1)}}_{\text{Oscillating in } R.\text{ Limit doesn't exist.}}$$

3 $\hat{a}(\xi) = e^{-10\xi^2}$. Calculate $a(x)$.

use inverse F.T. :

$$F^{-1}\{e^{-10\xi^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-10\xi^2} e^{ix\xi}\, d\xi$$

Some algebra:

$$-10\xi^2 + ix\xi = -10(\xi - ix/20)^2 - x^2/40$$

$$F^{-1}\{e^{-10\xi^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-10(\xi - ix/20)^2 - x^2/40}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/40}\int_{-\infty}^{\infty} e^{-10(\xi^2 - ix/20)^2}\, d\xi$$

Change of variable:

$$u = \sqrt{10}(\xi - ix/20), \quad d\xi = du/\sqrt{10}$$

$$\ldots = \frac{1}{\sqrt{2\pi}} e^{-x^2/40}\int_{-\infty - \sqrt{10}\,ix/20}^{\infty - \sqrt{10}\,ix/20} e^{-u^2}\, du/\sqrt{10}$$

$$= \frac{1}{\sqrt{20\pi}} e^{-x^2/40}\int_{-\infty}^{\infty} e^{-u^2}\, du \qquad \text{(By Cauchy's theorem)}$$

$$= \frac{1}{\sqrt{20\pi}} e^{-x^2/40}\sqrt{\pi} \qquad \text{(Gaussian Integral)}$$

$$= \frac{e^{-x^2/40}}{\sqrt{20}}$$

4 $b(x) = \begin{cases} 5 & \text{if } -1<x<3 \\ 0 & \text{else} \end{cases}$

$$\begin{aligned}
\sin(x) * b(x) &= b(x) * \sin(x) \\
&= \int_{-\infty}^{\infty} b(t-x)\sin(t)\,dt \\
&= \int_{-\infty}^{\infty} \left(\begin{cases} 5 & \text{if } -1<t-x<3 \\ 0 & \text{else} \end{cases} \right) \sin(t)\,dt \\
&= \int_{-\infty}^{\infty} \left(\begin{cases} 5 & \text{if } -1+x<t<3+x \\ 0 & \text{else} \end{cases} \right) \sin(t)\,dt \\
&= \int_{x-1}^{x+3} 5\sin(t)\,dt \\
&= -5\cos(x+3) + 5\cos(x-1)
\end{aligned}$$

5 In class, we calculated that

$$u(x,t) = u(x,0) * \frac{e^{-x^2/4\alpha t}}{\sqrt{4\pi\alpha t}}$$

solves the heat eq. with initial temperature $u(x,0)$. We have $\alpha=2$, so

$$\begin{aligned}
u(x,t) &= e^{-x^2/2} * \frac{e^{-x^2/8t}}{\sqrt{8\pi t}} \\
&= \int_{-\infty}^{\infty} e^{-(y-x)^2/2} \frac{e^{-y^2/8t}}{\sqrt{8\pi t}}\,dy \\
&= \frac{1}{\sqrt{8\pi t}} \int_{-\infty}^{\infty} e^{-y^2/2+xy-x^2/2-y^2/8t}\,dy \\
&= \frac{1}{\sqrt{8\pi t}} \sqrt{\frac{8\pi}{4+t}}\, e^{-tx^2/(8+2t)}
\end{aligned}$$

(I used Mathematica.)

6 Shannon sampling formula:

$$f(t) \approx \sum_{k=-\infty}^{\infty} f\left(\frac{2\pi}{T}k\right) \frac{\sin(Tt/2 - k\pi)}{Tt/2 - k\pi}$$

x	f(x)
-4	7
-2	1
0	18
2	0
4	0

use $\frac{2\pi}{T} = 2$

$\Longrightarrow$ $T = \pi$

$$f(t) \approx \sum_{k=-2}^{2} f(2k) \frac{\sin(\pi(t/2 - k))}{\pi(t/2 - k)}$$

$$= f(-4)\sin(\pi(t/2+2))/\pi(t/2+2)$$
$$+ \dots + f(4)\sin(\pi(t/2-2))/\pi(t/2-2)$$

$$= 7\frac{\sin(\pi(t/2+2))}{\pi(t/2+2)} + \frac{\sin(\pi(t/2+1))}{\pi(t/2+1))}$$
$$+ 18\frac{\sin(\pi(t/2))}{\pi t/2}$$